Characteristic Elastic Systems of Time-Limited Optimal Maneuvers

Arthur L. Hale* and Ronald J. Lisowski†
University of Illinois, Urbana-Champaign, Illinois

This paper extends earlier work by the authors on optimizing an elastic system and its active control. Maneuvers from an initial to a final state in a finite time interval are considered. An active generalized control force that accomplishes the desired maneuver of a prespecified system is optimal if it minimizes a given quadratic cost functional. By also varying a set of design parameters, the elastic system can be determined so as to further minimize the cost. Here, the elastic system that minimizes the actual control cost is compared with the system that minimizes the ratio of actual cost to the cost of optimally maneuvering a rigid system of the same inertial properties. It is shown that an elastic system corresponding to an extremum of the ratio is actually a characteristic of the time-limited maneuver. Because the time interval is fixed, a characteristic elastic system is tuned to the boundary conditions specified at the terminal times. The implication for rest-to-rest, spinup, and spin reversal maneuvers of spacecraft is that the optimal control for a characteristic elastic spacecraft is identical to the optimal control for the same spacecraft as if it were rigid.

I. Introduction

WHEN a structure such as a spacecraft is very flexible, its active control system can excite and otherwise significantly interact with its elastic deformations. Thus, the idea arises of achieving the best balance between a flexible structure's design and the energy required for its active control. To do this, one must first consider such questions as how to pose this problem mathematically, how to obtain a solution of the mathematical problem if one exists, whether or not solutions are numerically feasible in the case of actual complex structures, and ultimately what the solution means, i.e., what it tells us that we do not already know. With these questions in mind, particularly the last one, this paper culminates a series of works[1-5] that address the so-called "integrated optimization" of a structure and its active control.

In Ref. 1-5 and herein, the type of active control considered is one needed to perform a time-limited maneuver, that is, a controlled movement from a known initial state to a specified final state in a fixed time interval. The restriction to this type of control is justified for spacecraft because maneuvering loads are one of the most signficant sources of large-magnitude disturbances. Example sources of loads are thruster firings for station-keeping maneuvers and abrupt reaction wheel and/or thruster excitation for rapid large-angle rotational maneuvers. Note that these control actions necessarily affect the entire flexible structure because they require changing the motion of the rigid-body modes.

Our "integrated optimization" problem seeks to minimize the control energy required for a maneuver by varying both the time dependence of the control itself and a set of structural parameters representing quantities such as member dimensions. Mathematically, this is a parametric optimal control problem[6] with the cost functional being the control energy. By allowing the structural parameters to vary, a balance can be obtained between the inertia (e.g., mass) and the control cost of a maneuver. Note that while a rigid body's control cost decreases as the inertia is decreased, for a flexible structure the control cost depends not only on the inertia of the structure as if it were rigid but also on the effects of flexibility. For example, minimizing mass can lead to a very flexible structure and, although its inertia is relatively small, maneuvering the structure may require exorbitant control energy because control forces of large magnitude and rapid variation are needed to control the elastic deflections. This fact is graphically illustrated in Sec. III.

The main new contribution of this paper is that solutions of the parametric optimal control problem minimizing actual control cost are now compared with solutions that minimize the ratio of actual control cost to the cost of optimally controlling a rigid body of the same inertial properties. When minimizing the latter ratio of costs, a characteristic value problem arises. Hence, its solutions are referred to as characteristic elastic systems for the time-limited optimal maneuver in question. It is shown that in an important special case minimizing the ratio degenerates into a much simpler natural frequency placement problem that tunes the structure to a time-limited maneuver.

Finally, we note that a slightly different approach than that of Refs. 1-5 for the "integrated optimization" problem concerning maneuvers has been presented in Refs. 7 and 8. In addition, a number of recent papers[9-14] have addressed in one way or another the related integrated optimization problem for vibration regulation. Vibration regulation is not addressed here in order to limit the paper's scope.

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*Assistant Professor, Department of Aeronautical and Astronautical Engineering, appointed jointly to the Civil Engineering Department (presently Senior Engineering Specialist, Space Systems Div., General Dynamics Corp., San Diego, Calif.). Member AIAA.

†Major, USAF, Department of Aeronautical and Astronautical Engineering (presently Assistant Professor and Deputy for Labs and Research, Department of Astronautics, U.S. Air Force Academy, Colorado, Springs, Colo.). Member AIAA.

II. Integrated Structure/Control Optimization—Problem Statements

Minimizing Actual Control Cost

We consider the specified terminal value maneuver of a flexible structure in the specified finite time interval $0 \leq t \leq t_f$. In this subsection, a control vector $F(t) \in R^{N_c}$ and a vector of structural parameters $\xi \in R^{N_d}$ are desired that minimize the cost functional

$$J(F,\xi) = \int_0^{t_f} ½(F^T RF)\mathrm{d}t \tag{1}$$

subject to the following differential equation constraints, specified initial values, specified final values, and inequality constraints, respectively:

$$M(\xi)\ddot{U}(t) + K(\xi)U(t) = GF(t) \tag{2a}$$

$$U(0) = U_0 \tag{2b}$$

$$\dot{U}(0) = V_0 \tag{2c}$$

$$U(t_f) = U_f \tag{2d}$$

$$\dot{U}(t_f) = V_f \tag{2e}$$

$$\xi_i \geq 0, \qquad i = 1,2,\ldots,N_d \tag{2f}$$

In Eq. (1), R is an $N_c \times N_c$ positive definite symmetric weighting matrix, and in Eqs. (2) $U(t) \in R^N$ is the generalized displacement vector and $M(\xi)$ and $K(\xi)$ are the $N \times N$ symmetric mass matrix and stiffness matrix, respectively. It is assumed that the actuator positions are not affected by changes in the structural parameters. Therefore, while the matrix G depends on the actuator positions it does not depend on ξ. Clearly, this is a convenient simplifying assumption that can be relaxed later at the expense of an increase of complexity. The constraints in Eq. (2f), where ξ_i is the ith entry of ξ, are intended to represent non-negativity constraints on physical parameters such as structural member widths, lengths, and/or cross-sectional areas. In many problems, the lower bound of zero might be replaced by a positive value representing a minimum gage for structural members. For simplicity, other explicit constraints are not considered, although in the future one might want to consider bounds on dynamic stresses, for example. The specified terminal values in Eqs. (2b-2e) define the desired maneuver and Eq. (2a) is assumed to be obtained by discretizing the equations of motion of an actual distributed structure by the finite element method. Of course, the modeling methodology is critically important to the integrated problem for actual structures, although it is not addressed explicitly here.

Note that we do not consider a penalty on the elastic deflections in the cost, Eq. (1). This is specifically because it is very difficult to assign a priori a meaningful relative penalty on the deflection vs the penalty on control energy. It is especially true of the integrated problem wherein the structural parameters are varied; it is not so difficult in the separate optimal control for a given structure. Moreover, in actual design, bound-type constraints on stresses and/or deflections are given. Due to the difficulties of including such dynamic constraints, e.g., disjoint feasible regions,[15,16] their inclusion is left to future work.

The necessary conditions are derived by the calculus of variations and their derivation is discussed in Ref. 4. The differential equation constraints (2a) are invoked by introducing a vector U^* of adjoint displacements (Lagrange multipliers) and appending the constraints to Eq. (1). After doing so, the variations of F, U, and U^* are arbitrary so that their coefficients all vanish, leading to the $2N + N_c$ necessary conditions for an optimal control

$$F(t) = R^{-1}G^T U^*(t) \tag{3a}$$

$$M(\xi)\ddot{U}(t) + K(\xi)U(t) = GF(t) \tag{3b}$$

$$M(\xi)\ddot{U}^*(t) + K(\xi)U^*(t) = 0 \tag{3c}$$

When a parameter is on the boundary of the admissible parameters, the variation of that particular parameter must itself vanish. However, in the neighborhood of extrema not on the boundary, the coefficient of the variation of each ξ_i vanishes, leading to the N_d necessary conditions

$$L_i = \int_0^{t_f} U^{*T}\left(\frac{\partial M}{\partial \xi_i}\ddot{U} + \frac{\partial K}{\partial \xi_i}U\right)\mathrm{d}t = 0, \quad i = 1,2,\ldots,N_d \tag{3d}$$

The necessary conditions [Eqs. (3)] are a hybrid system of nonlinear equations that must be solved numerically.[2,3,5] For specified parameters ξ, Eqs. (3a-3c) are linear. In conjunction with Eqs. (2b-2e), they represent a two-point boundary value problem for an optimal control. The linear optimal control problem is readily solved by standard techniques (e.g., Ref. 6) after first converting Eqs. (3a-3c) to a coupled system of homogeneous first-order state-costate equations.

Once the optimal control problem is solved, the N_d entries L_i of the gradient of J with respect to ξ can be evaluated. One method for doing this numerically is to evaluate the optimal trajectory at many discrete times and use these values to compute the integrals in Eq. (3d) as well as in Eq. (1) via Simpson's rule. The gradient can then be used in a projected gradient iterative algorithm for determining an admissible minimizing value of the parameter vector. Of course, such an algorithm converges only to local minima and ensuring that one obtains a global minimum is plagued with difficulties due to the fact that J is not globally convex. Also, because the numerical solution for even relatively simple structures can be computationally expensive, it is natural to consider representing the flexible structure by a truncated number $n\,(n \leq N)$ of its lowest modes of free vibration.[3-5,7,8] The parameter optimization is then based on a reduced-order model of dimension n.

Note that the present work is concerned with the solution itself and not specifically with an algorithm for numerically computing the solution. Therefore, details of a numerical procedure are omitted. Nevertheless, as the work of Refs. 1-5 and 7-14 attests, algorithms are available although they may not yet be efficient for large problems. A spcialized computer program produced the numerical results herein. Details regarding the numerical procedure, as well as details about the use of reduced-order models can be found in Refs. 3-5.

Minimizing the Ratio of Actual Control Cost to Rigid-Body Control Cost

Next, for comparison, we consider a control vector $F(t) \in R^{N_c}$ for the actual elastic system, a control vector $F_R(t) \in R^{N_c}$ for a rigid body of the same inertial properties, and a vector of structural parameters $\xi \in R^{N_d}$. Values of the parameters are desired that minimize a functional that is the ratio of cost functionals for the actual system and for the rigid body, namely,

$$\lambda = \frac{J(F,\xi)}{J_R(F_R,\xi)} = \int_0^{t_f} ½\,(F^T RF)\mathrm{d}t \bigg/ \int_0^{t_f} ½\,(F_R^T RF_R)\mathrm{d}t \tag{4}$$

Note that the same weighting matrix R appears in both the numerator and the denominator of Eq. (4).

The minimization of λ is subject to the constraints of Eqs. (2a-2e) for the actual structure, the inequality constraints of Eqs. (2f) as well as the following differential equation constraints and specified initial and final values for the rigid body:

$$m_R(\xi)\ddot{u}_R(t) = g_R F_R(t) \tag{5a}$$

$$u_R(0) = u_{R0} \tag{5b}$$

$$\dot{u}_R(0) = v_{R0} \tag{5c}$$

$$u_R(t_f) = u_{Rf} \tag{5d}$$

$$\dot{u}_R(t_f) = v_{Rf} \tag{5e}$$

In Eqs. (5a-5e), $u_R(t) \in R^{n_R}$ is the generalized displacement vector for the rigid body and $m_R(\xi)$ is the $n_R \times n_R$ symmetric mass matrix. The specified terminal values [Eqs. (5b-5e)] define the desired rigid-body maneuver. Moreover, Eq. (5a) is obtained from Eq. (2a) by transforming to a set of reduced coordinates that span the space of rigid-body modes. For simplicity, we make the assumption that by varying the parameters no additional rigid-body degrees of freedom are introduced. This is the equivalent to excluding the rigid-body modes that arise when, for example, parameters vanish and the structure becomes two or more disjoint structures, each with its own set of rigid-body modes. Therefore, the space containing the rigid-body modes is constant and a basis for the space that does not depend on the parameters ξ is justified. The transformation itself can then be preformed by constraining U to be a linear combination of n_R constant vectors φ_{Ri} that span the space of rigid-body modes, i.e., by writing

$$U(t) = \sum_{i=1}^{n_R} \varphi_{Ri} u_{Ri}(t) = \Phi_R u_R(t) \tag{6}$$

By definition, the vectors φ_{Ri} have the property $K\varphi_{Ri} = 0$, which holds for all admissible values of ξ even though K depends on ξ and φ_{Ri} does not. Hence, upon substituting Eq. (6) into Eq. (2a) and premultiplying by Φ_R^T, one obtains Eq. (5a) where

$$m_R(\xi) = \Phi_R^T M(\xi) \Phi_R \tag{7a}$$

$$g_R = \Phi_R^T G \tag{7b}$$

Finally, the ratio of Eq. (4) is meaningful only if the maneuver of the actual flexible structure is intimately related to the maneuver of its corresponding rigid model. Our practical interest is in maneuvers of the flexible structure for which the terminal conditions [Eqs. (2b-2e)] specify rigid-body motions only, so that elastic deflections and velocities are zero at both the initial and final times, i.e., at the terminal times. Then, the terminal conditions [Eqs. (5b-5e)] for the rigid body are a projection of Eqs. (2b-2e). One possible projection is to use $u_{R0} = (\Phi_R^T M \Phi_R)^{-1} \Phi_R^T M U_0$ and a similar expression for each of Eqs. (5c-5e). However, because the rigid-body modes are constant, they need not be orthogonal with respect to $M(\xi)$ for all (or even any) value of ξ. Therefore, it is more convenient in practice to consider the constant projection $u_{R0} = (\Phi_R^T \Phi_R)^{-1} \Phi_R^T U_0$ and similarly for Eqs. (5c-5e). Of course, for a rigid body the elastic deflections are zero for all time and the orthogonal complement of Eqs. (5b-5e) for the elastic modes is, therefore, trivially satisfied.

Again the necessary conditions are derived by the calculus of variations. The differential equation constraints (2a) and (5a) are invoked by introducing U^* and u_R^* and appending the constraints to the numerator and denominator of Eq. (4), respectively. Taking the variation leads to Eqs. (3a-3c) for the optimal control of the actual structure, to the $2n_R + N_c$ necessary conditions for the optimal control of the corresponding rigid body, namely,

$$F_R(t) = R^{-1} g_R^T u_R^*(t) \tag{8a}$$

$$m_R(\xi) \ddot{u}_R(t) = g_R F_R(t) \tag{8b}$$

$$m_R(\xi) \ddot{u}_R^*(t) = 0 \tag{8c}$$

and, finally, to the N_d necessary conditions (for extrema not on the boundary of admissible parameters) for the parameters ξ_i

$$L_{\lambda i} = \int_0^{t_f} U^{*T}\left(\frac{\partial M}{\partial \xi_i}\ddot{U} + \frac{\partial K}{\partial \xi_i}U\right)\mathrm{d}t$$

$$-\lambda \int_0^{t_f} u_R^{*T} \frac{\partial m_R}{\partial \xi_i} \ddot{u}_R \mathrm{d}t = 0, \qquad i = 1,2,\ldots,N_d \tag{9}$$

Note that although M and m_R vary with ξ, when using a constant projection the terminal values [Eqs. (5b-5e)] do not vary and no additional terms appear in the necessary conditions [Eqs. (8a-8c) and (9)].

The function λ is a classic form, namely, the ratio of two positive functions of ξ. It is, therefore, no surprise that the conditions of Eq. (9) have the classic form of a characteristic value problem. Clearly, the best we can do is to have the cost for the flexible structure be equal to the cost for the same structure as if it were rigid. The characteristic values of λ are, therefore, all greater than or equal to one. If one assumes that the "spectrum" of characteristic values is discrete (no proof to this effect is available and a continuous spectrum may exist), then corresponding to each discrete value of $\lambda^{(r)} \geq 1$ is a characteristic vector of structural parameters $\xi^{(r)}$ $(r = 1,2,\ldots)$. The parameter vectors correspond to local minima of Eq. (4) and their actual values for most elastic systems must be found numerically.

Solving the general characteristic value problem above is difficult and no attempt has been made to provide a general solution algorithm. Instead, the sequel concentrates on the special case when discrete λ equal to one occur. This case is of special interest and significance because it represents an ideal situation that the general case tends toward. In the next section, it will be seen that a sufficient condition for it is that all of a structure's natural frequencies be assignable by choosing feasible values of the structural parameters. In this special case, the problem of minimizing the ratio degenerates into the much simpler problem of placing the natural frequencies.

III. Simple Two-Degree-of-Freedom Example

We demonstate the behavior of the ratio λ and its relationship to J in this section by considering a simple example, namely, a rest-to-rest translational maneuver of two equal masses of mass m connected by a uniform elastic bar. The bar is idealized as a single finite element in axial extension. It has the mass per unit of volume γ, length L, modulus of elasticity E, and cross-sectional area A. The uniform area A is free to be determined. The maneuver is to be accomplished with only a single control force acting at one of the end masses. Hence, the cross-sectional area A cannot become zero without losing the ability to translate the second mass.

The equations of motion for this system have the form of Eq. (2a) with $N=2$, $N_c=1$, $N_d=1$, $U=\{U_1, U_2\}^T$, $\xi_1 = A$,

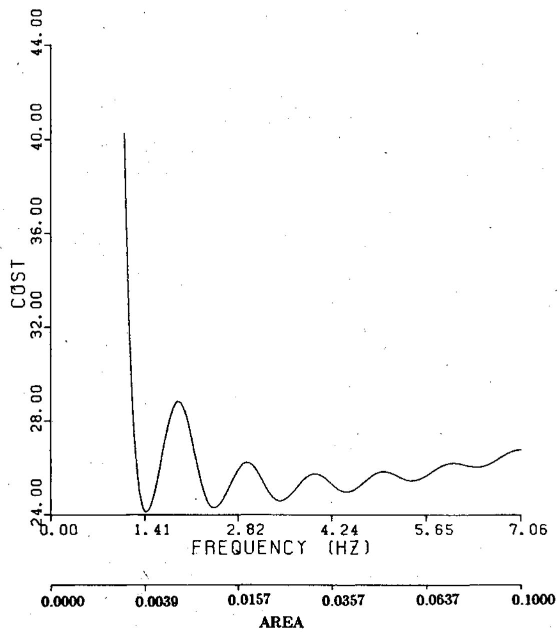

Fig. 1 Actual control cost vs frequency, simple example.

$$M(A)=\begin{bmatrix} m+\frac{\gamma L}{3}A & \frac{\gamma L}{6}A \\ \frac{\gamma L}{6}A & m+\frac{\gamma L}{3}A \end{bmatrix} \tag{10a}$$

$$K(A)=\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad G=\begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \tag{10b,c}$$

Due to symmetry, the eigenvectors (not normalized) are always

$$\varphi_R=\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}, \varphi_E=\begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \tag{11a,b}$$

where φ_R is the rigid-body mode and φ_E the elastic mode. The equations of motion can be decoupled by transforming to modal coordinates. The decoupled equations again have the form of Eq. (2a) except now with $U=\{u_R,u_E\}^T$ and

$$M(A)=\begin{bmatrix} 2m+\gamma LA & 0 \\ 0 & 2m+\frac{\gamma LA}{3} \end{bmatrix} \tag{12a}$$

$$K(A)=\begin{bmatrix} 0 & 0 \\ 0 & \frac{4EA}{L} \end{bmatrix} \tag{12b}$$

$$G=\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \tag{12c}$$

A control force F, a scalar in this problem, is desired that translates the system between the following terminal conditions (given in term of the modal coordinates)

$$U_0=0, \quad V_0=0, \quad U_f=\begin{Bmatrix} 1 \\ 0 \end{Bmatrix}, V_f=0 \tag{13a-d}$$

and that minimizes Eq. (1) with $R=1$. The terminal conditions [Eqs. (13)] specify a rest-to-rest translational maneuver. Because the equations of motion are decoupled, the optimal control, the modal displacements, and the modal velocities can be obtained earily. In terms of four constants $C_1,\ldots, C_4$ depending on the final conditions (13c,d), one obtains

$$F=(C_1+C_2t+C_3\cos\omega t+C_4\sin\omega t) \tag{14}$$

Table 1 Comparison of control cost minima for the simple example

Minima of the cost ratio		Minima of the actual cost		
Area	Freq, Hz	Area	Freq, Hz	Cost
0.00404	1.4303	0.00403	1.4293	24.097
0.01196	2.4590	0.01193	2.4542	24.287
0.02387	3.4709	0.02369	3.4575	24.574
0.03983	4.4774	0.03933	4.4489	24.959
0.05990	5.4815	0.05876	5.4297	25.445
0.08416	6.4844	0.08190	6.3979	26.034

$$u_R=\frac{1}{(2m+\gamma LA)}\left[C_1\frac{t^2}{2}+C_2\frac{t^3}{6}+\frac{C_3}{\omega^2}(1-\cos\omega t) + \frac{C_4}{\omega^2}(\omega t-\sin\omega t)\right] \tag{15a}$$

$$u_E=\frac{1}{[2m+(\gamma L/3)A]}\left[\frac{C_1}{\omega^2}(1-\cos\omega t)+\frac{C_2}{\omega^3}(\omega t-\sin\omega t) + \frac{C_3}{2\omega}t\sin\omega t+\frac{C_4}{2\omega^2}(\sin\omega t-\omega t\cos\omega t)\right] \tag{15b}$$

$$\dot{u}_R=\frac{1}{(2m+\gamma LA)}\left[C_1t+C_2\frac{t^2}{2}+\frac{C_3}{\omega}\sin\omega t + \frac{C_4}{\omega}(1-\cos\omega t)\right] \tag{15c}$$

$$\dot{u}_E=\frac{1}{[2m+(\gamma L/3)A]}\left[\frac{C_1}{\omega}\sin\omega t+\frac{C_2}{\omega^2}(1-\cos\omega t) + \frac{C_3}{2\omega}(\omega t\cos\omega t+\sin\omega t)+\frac{C_4}{2}t\sin\omega t\right] \tag{15d}$$

where the natural frequency of the elastic mode is

$$\omega=\left(\frac{12EA}{6mL+\gamma L^2A}\right)^{1/2} \tag{16}$$

Of course, the four specified initial conditions [Eqs. (13a) and (13b)] are already reflected in Eqs. (15) and, as already mentioned, the four specified final conditions [Eqs. (13c) and (13d)] yield four equations for the constants C_1, C_2, C_3, and C_4. It is easiest to compute the constants numerically. However, the control cost J can be evaluated analytically in terms of the constants. The resulting expression is evaluated numerically once the constants are known. One can also easily determine the following optimal control and optimal control cost, respectively, for the rigid body of mass $(2m+\gamma LA)$:

$$F_R=\frac{6}{t_f^2}(2m+\gamma LA)\left(1-\frac{2t}{t_f}\right) \tag{17a}$$

$$J_R=6\left(\frac{1}{t_f^3}\right)(2m+\gamma LA)^2 \tag{17b}$$

The ratio of Eq. (4) is, therefore, readily computed for comparison to J.

The actual control cost J is plotted vs frequency $(\omega/2\pi)$ in Fig. 1 for the specific values $t_f=1$, $m=1$, $\gamma=1$, $L=1$, and $E=1{,}000$. The area A is also shown, although the frequency is not a linear function of A. The chosen values of E, m, L, and γ always allow a great deal of freedom in assigning ω by varying A. Note the many local minima in Fig. 1 and the trend toward increasing cost with structural frequency. Also, note the very high control cost associated with low structural frequencies. Although the entire structure's mass decreases with decreasing structural frequency, the control cost actually increases dramatically. At low structural frequencies, the mass must be accelerated and decelerated rapidly in order to control the second mass via the flexible connecting bar. This graphically illustrates the tradefoff between mass and flexibility mentioned in the introduction.

The numerical values of frequencies and areas of the minima of J are shown in Table 1, along with the corresponding values of the cost J. For comparison, the ratio λ is plotted in Fig. 2 and the frequencies and areas of the minima

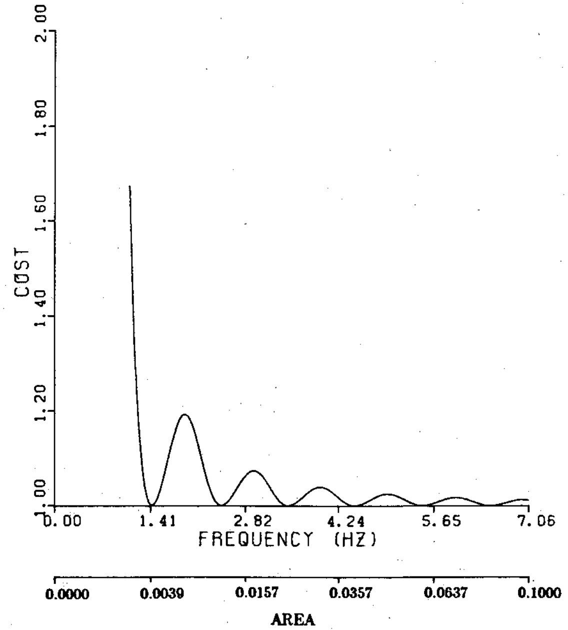

Fig. 2 Ratio of control costs vs frequency, simple example.

are also shown in Table 1. Note that the minima of the ratio are all equal to unity. Moreover, the minima of the ratio occur at nearly the same frequencies (areas) as the minima of the actual control cost. The value of the actual optimal control cost is very slightly higher for a value of A that corresponds to a minimum of λ rather than a minimum of J. However, the price in choosing a value that corresponds to a minimum of λ is likely worth paying, particularly because of the meaning of the minima of λ. Finally, note that the ratio approaches unity everywhere as the structure is stiffened. This is consistent with the fact that flexibility effects are relatively unimportant when maneuvering stiff structures. In fact, we propose that the ratio λ is a meaningful dimensionless measure of the importance of flexibility.

To see the further meaning of the minima of λ, let us write the rigid-body control force [Eq. (17a)] as $F_R = \alpha(1-2t/t_f)$ and apply this force to the actual flexible structure. The equation of motion for the elastic mode has the form

$$\ddot{u}_E + \omega^2 u_E = b\left(1 - \frac{2t}{t_f}\right) \tag{18}$$

where b is the appropriate participation of the force in the modal equation. Note that ω and b both depend on A, the design parameter. Using the initial conditions $u_E(0) = \dot{u}_E(0) = 0$, the solution of Eq. (18) is

$$u_E(t) = \frac{2b}{\omega^3 t_f}\sin\omega t - \frac{b}{\omega^2}\cos\omega t + \frac{b}{\omega^2}\left(1 - \frac{2t}{t_f}\right) \tag{19}$$

Our desire is that the modal displacement and velocity are quiescent at the final time, i.e., $u_E(t_f) = \dot{u}_E(t_f) = 0$. These final conditions are generally not met because the force F_R is generally not a control force for the elastic body. However, the possibility exists that, by choosing ω (equivalently A), the final conditions can be met. Two equations for ω result by invoking the final conditions, namely,

$$\frac{2}{\omega^3 t_f}\sin\omega t_f - \frac{1}{\omega}\cos\omega t_f - \frac{1}{\omega^2} = 0 \tag{20a}$$

$$\frac{2}{\omega^2 t_f}\cos\omega t_f + \frac{1}{\omega}\sin\omega t_f - \frac{2}{\omega^2 t_f} = 0 \tag{20b}$$

Table 2 Characteristic natural frequencies for rest-to-rest maneuver

Mode no.	Natural frequency, Hz	Mode no.	Natural frequency, Hz
0	0.0000	13	13.4925
1	1.4303	14	14.4930
2	2.4590	15	15.4935
3	3.4709	16	16.4939
4	4.4774	17	17.4942
5	5.4815	18	18.4945
6	6.4844	19	19.4948
7	7.4965	20	20.4951
8	8.4881	21	21.4953
9	9.4893	22	22.4955
10	10.4903	23	23.4957
11	11.4912	24	24.4959
12	12.4919	25	25.4960

Equations (20) are transcendental equations in terms of ω and values of ω that satisfy both equations simultaneously can be found numerically. Although roots do exist for one equation that are not roots of the other equation, an infinite number of roots to both equations simulataneously also exist.

The first 25 simultaneous roots of Eq. (20) i.e., the frequencies ($\omega/2\pi$), are listed in Table 2 for $t_f = 1$. Comparing the frequencies in Table 2 with frequencies of minima of λ reveals that the roots of Eqs. (20) occur at precisely the minima of λ, i.e., at characteristic frequencies. *Therefore, the roots correspond to an infinite number of characteristic natural frequencies.*

The implication is that the optimal control for an elastic system with all natural frequencies at values of ω satisfying Eqs. (20) is the same as the optimal rigid-body control. Here, the rigid-body control performs a rest-to-rest maneuver. The rigid-body control for other maneuvers, such as speedup and speed reversal maneuvers, yields different characteristic natural frequencies. For example, the optimal rigid-body control force for a speedup or speed reversal maneuver with initial position specified and final position free is simply a constant value b for $0 \le t \le t_f$ and zero for $t > t_f$ or $t < 0$. Then, the conditions analogous to Eqs. (20) are

$$(1/\omega)(1 - \cos\omega t_f) = 0 \tag{21a}$$

$$\sin\omega t_f = 0 \tag{21b}$$

and the characteristic frequencies correspond to $\omega t_f = 2i\pi$, ($i = 1,2,\ldots$) or $\omega/2\pi = i/t_f$. Whereas a characteristic elastic system depends on the specific maneuver, note that Eqs. (20) and (21) do not depend on the magnitude of the maneuver, which would be reflected in the actual value of b. Hence, a characteristic elastic system is valid for both large and small maneuvers performed in the same specified time interval.

Note that the same values of natural frequency (equivalently, structural parameters) have been obtained here by two totally different approaches. The second approach provides an interpretation of the first. It shows that a sufficient condition for discrete minima of the ratio λ to be equal to one is for all of a structure's natural frequencies to be independently assignable by varying the structural parameters. In this case, the problem of minimizing λ corresponds to a natural frequency assignment problem. When all of the natural frequencies cannot be assigned independently, the minima of λ will be greater than one by an amount that depends on the ability to "tune" those modes requiring the most energy to control. This simple example also shows that the general problem of minimizing J gives an optimal structure that is nearly the same as a characteristic structure, thus the physical meaning of an optimal structure is now better understood.

Of course, a structure composed of two masses connected by a uniform elastic bar is not representative of an actual complex structure or, in particular, a large space structure. Moreover, the concept of a characteristic elastic system for the maneuver is degenerate when only one elastic mode is present. The real allure of the concept is in placing natural frequencies of all elastic modes (or at least a number of the lowest modes) of a complex structure at values satisfying Eqs. (20), for example. In complex systems with many elastic modes, the placement of natural frequencies at the characteristic values may not always be possible and having values in order of Table 2 for a rest-to-rest maneuver is also not necessary.

IV. Single-Axis Rotational Maneuver of a Symmetric Four-Boom Structure

We now consider a single-axis rest-to-rest rotational maneuver of an idealized four-boom flexible structure (Fig. 3). This structure is more representative because many elastic modes are present. The booms are identical and only antisymmetric bending deflections are allowed. A control torque $f_1(t)$ acts on the rigid hub. Although shown in Fig. 3, the boom torques f_2 are not used here. Therefore, the booms are controlled only by moving the hub. Each boom is modeled by four finite elements of equal length, constant thickness, and variable width (Fig. 4). Thus, the structural design parameters are ξ_1, ξ_2, ξ_3, and ξ_4, where ξ_1 is the root width and ξ_4 is the tip width. Equations of motion are given in Refs. 2, 3, and 5. Because the booms are in bending due to the applied torque, the structure is uncontrollable when the root width becomes zero. A singular case also occurs when the entire mass of the booms is zero, as opposed to booms with infinitesimally small mass. We assume that some mass, however small, distributed along the booms is essential, thus ruling out this singular case. The full dimension of the discrete equations is $N=9$. The same rest-to-rest maneuver and parameters as in Refs. 3 and 5 are used here. Specifically, the structure is slewed from rest and $\theta=0$ to rest and $\theta=1$ in one unit of time using only the hub torque. The booms' elastic deformations are quiescent initially and specified to be quiescent at the final time. In the results that follow, the parameter optimization associated with minimizing Eq. (1) is based on a reduced-order model that contains only the first four ($n=4$) natural modes of free vibration.[3-5] The modes are updated at each change of the parameters ξ, although extrapolation formulas could be used (e.g., as in Ref. 7). Of course, when the optimal control for the reduced-order model is applied to the full-order model, the residual modes are also excited. Because the residual modes are not considered in calculating the control, their final displacements and velocities are not generally quiescent.

First, we consider minimizing the actual control cost J for the maneuver. The optimal parameters (in the sense of minimizing J) are presented in Table 3. For comparison, a nonoptimal set of constant parameters, each of which is the average value of the optimal parameters, is also included. The natural frequencies corresponding to the eigenvalues used in the reduced-order model are listed, along with the natural frequencies corresponding to the residual modes. The table also includes the final values for the residual mode displacements and velocities for both cases. The residual mode responses result from taking the optimal control for the reduced-order model and applying it to the full-order model. Time histories obtained from the full-order simulations are shown in Figs. 5 and 6 for the optimal and non-optimal cases, respectively. Figure 7 presents the magnitude spectrum of the control for the two cases. The optimal parameter case is shown in solid lines and the nonoptimal parameter case in dashed lines. Many attractive properties are observed as a consequence of choosing optimal parameters. For example, the optimal control for the optimal structure is both lower in magnitude and has less excitation at high frequencies than the optimal control for the structure with nonoptimal parameters. The tip deflections and velocities are also much lower for the optimal structure. This demonstrates the benefits obtainable by performing the integrated optimization. Because the parameters are determined so as to minimize J, however, the structure is not tuned exactly to the maneuver. This is seen by observing the full-order simulation results in Table 4, where an optimal control for the corresponding rigid-optimal and rigid-nonoptimal structure is applied to the full-order model of the structure. While the modal excitation at the final time for the optimal structure is much less than that for the nonoptimal structure, its level is significant.

Next, a characteristic structure corresponding to the same rest-to-rest maneuver is considered for comparison. By choosing the $N_d=4$ parameters ξ_i, only four elastic-body natural frequencies can be assigned. The rigid-body mode is always present, and hence $n=5$ natural frequencies can be chosen to satisfy Eqs. (20). However, it is not possible in this example to assign the first five frequencies in order of Table 2. This is due to the coarse structural discretization in which only four finite elements of equal length and uniform width are used. Instead, we assign the frequency of the first elastic mode to be

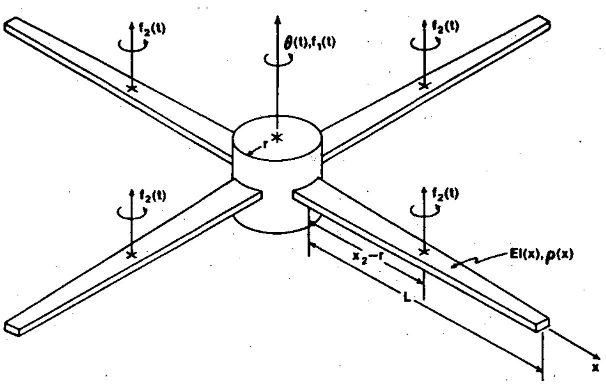

Fig. 3 Undeformed symmetric structure.

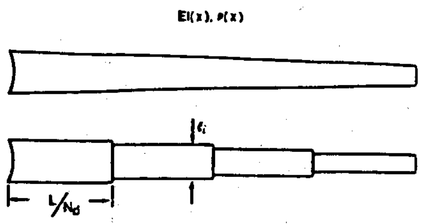

Fig. 4 Finite element model of flexible boom.

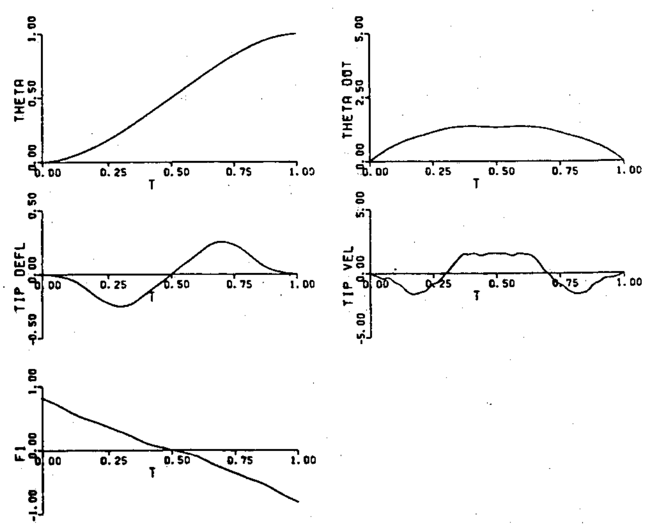

Fig. 5 Time histories for the optimal structure ($N_d=4$, $n=4$).

$\omega_2/2\pi = 1.4303$. The remaining freedom in choosing the parameters allows assigning $\omega/_3 2\pi = 3.4709$, $\omega/_4 2\pi = 7.4865$, and $\omega/_5 2\pi = 17.4942$. The parameters are presented in Table 5 along with all associated natural frequencies. For the four-parameter case considered here, values of the parameters to assign characteristic frequencies can be determined by trial and error, although many simple numerical algorithms to accomplish the same goal can be concocted. As mentioned previously, the interest here is in the solution itself and not in algorithms for computing the solution numerically.

Because the first five natural frequencies are characteristic frequencies for the maneuver, the optimal control for the rigid body does a remarkably good job maneuvering the actual flexible structure. Table 5 presents the final values of modal displacement and velocity for all elastic modes when the rigid-body optimal control torque is applied. Because the rigid-body mode is the only mode explicitly controlled, all other modes are termed residual modes in Table 5. Note that the level of excitation at $t = t_f = 1$ is small for all residual modes. It is slightly higher for modes 6-9 because the frequencies of these modes were not explicitly assigned. Figure 8 presents the time histories for the maneuver of the tuned structure, i.e., the *characteristic* structure. The optimal control is computed based on all nine modes. For comparison, Fig. 9 presents time histories for the full-order simulation when the optimal control for the rigid body is applied to the actual flexible structure. There is no noticeable difference between the two figures. The magnitude spectrum of the optimal control based on all nine modes is plotted as a solid line in Fig. 10. Also shown in Fig. 10 is the magnitude spectrum for the rigid-body control (a dased line). The first noticeable difference between the two occurs at frequencies above 20 Hz, where the natural frequencies were not assigned.

An additional interpretation of the characteristic frequencies of Table 2 is now noted. Each value of Table 2 corresponds precisely to one of the notch frequencies in the magnitude spectrum of the optimal rigid-body control (see Fig. 10).

Finally, it is interesting to observe the difference between the natural frequencies obtained by minimizing J and those shown in Table 2. The difference is small in the four-parameter case and decreases up to a point as the number of parameters is increased (Table 6). Moreover, as the number of parameters is increased, it is possible to obtain frequencies near more of the lowest values in Table 2. This is seen in Table 6, where 8 and 16 parameter results are displayed. As an aside, we point out that obtaining the optimal parameter values of Table 6 is complicated by the nonconvexity of the control cost. To obtain a global minimum, we solve a series of problems, the first problem having only one ($N_d = 1$) parameter (finite element) and the succeeding problems having $N_d = 2$, $N_d = 4$, $N_d = 8$, and finally $N_d = 16$ parameters. A global minimum of the one-parameter case is relatively easy to obtain. In each succeeding case, the solution of the previous problem (with $N_d/2$)

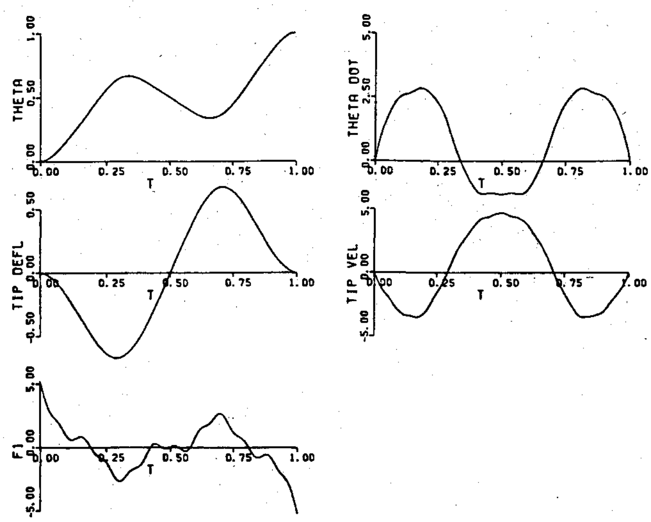

Fig. 6 Time histories for the nonoptimal structure ($N_d = 4$, $n = 4$).

Table 3 Parameters, frequencies, and residual response of four-boom structure ($N_d = 4$, $n = 4$)

Design parameters	Optimal 0.07533, 0.04065, 0.01837, 0.00723			Nonoptimal 0.03539		
Modeled natural frequencies, Hz	0, 1.396, 3.434, 7.508			0, 778, 3.972, 11.097		
Residual mode	Natural freq, Hz	Final displacement	Final velocity	Natural freq, Hz	Final displacement	Final velocity
5	17.354	0.9×10^{-5}	-0.2×10^{-2}	21.864	0.5×10^{-4}	0.3×10^{-2}
6	23.558	0.6×10^{-6}	0.5×10^{-3}	40.650	-0.3×10^{-5}	-0.1×10^{-2}
7	40.627	-0.4×10^{-6}	-0.2×10^{-3}	65.279	0.1×10^{-5}	-0.6×10^{-3}
8	72.831	-0.7×10^{-6}	-0.2×10^{-4}	103.480	-0.2×10^{-6}	0.2×10^{-4}
9	141.034	-0.2×10^{-6}	0.2×10^{-4}	169.779	-0.4×10^{-7}	0.4×10^{-4}

Table 4 Residual response of four-boom structure with rigid-body optimal control ($N_d = 4$, $n = 1$)

Design parameters	Optimal 0.07533, 0.04065, 0.01837, 0.00723			Nonoptimal 0.03539		
Residual mode	Natural frequency, Hz	Final displacement	Final velocity	Natural frequency, Hz	Final displacement	Final velocity
2	1.396	-0.6×10^{-3}	0.2×10^{-1}	0.778	0.5×10^{-1}	0.3×10^{1}
3	3.434	0.7×10^{-4}	-0.7×10^{-2}	3.972	0.6×10^{-3}	0.1×10^{-1}
4	7.508	-0.8×10^{-6}	-0.1×10^{-2}	11.097	0.4×10^{-4}	0.2×10^{-2}
5	17.354	0.8×10^{-5}	-0.2×10^{-2}	21.864	0.5×10^{-5}	0.5×10^{-3}
6	23.558	0.6×10^{-6}	0.5×10^{-3}	40.650	0.9×10^{-6}	0.2×10^{-3}
7	40.627	-0.4×10^{-6}	-0.2×10^{-3}	65.279	0.3×10^{-6}	0.9×10^{-4}
8	72.831	-0.6×10^{-6}	-0.2×10^{-3}	103.480	0.7×10^{-7}	0.4×10^{-4}
9	141.034	-0.2×10^{-6}	0.2×10^{-4}	169.779	0.6×10^{-8}	0.5×10^{-5}

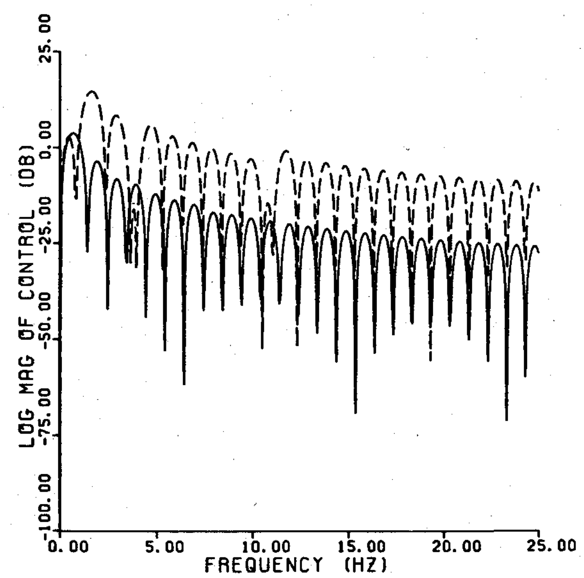

Fig. 7 Magnitude spectra of optimal control ($N_d = 4$, $n = 4$).

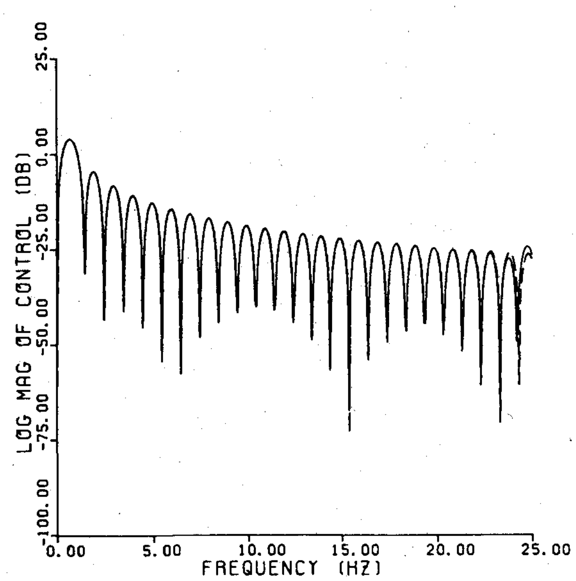

Fig. 10 Magnitude spectra of optimal controls for the charateristic structure ($N_d = 4$).

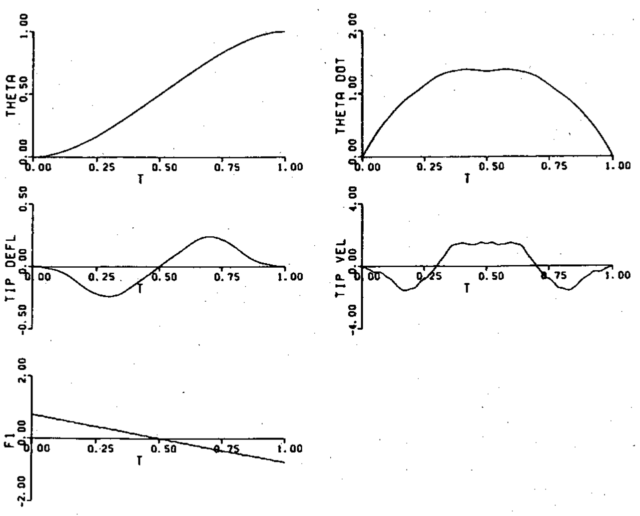

Fig. 8 Optimal control time histories for the characteristic structure ($N_d = 4$, $n = 9$).

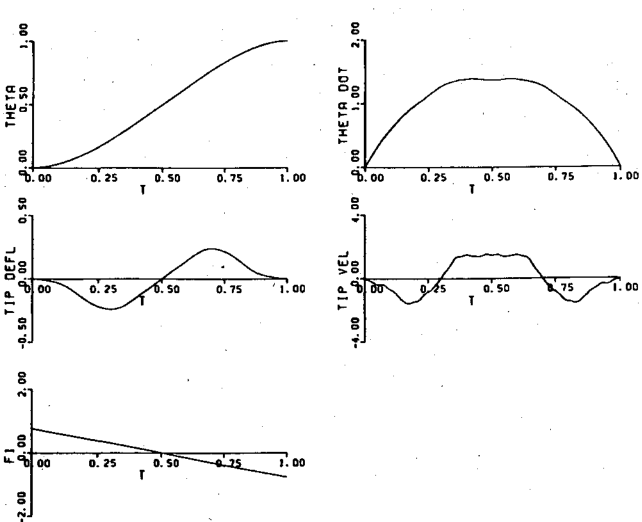

Fig. 9 Time histories for the rigid-body optimal control applied to the characteristic structure ($N_d = 4$).

Table 5 Parameters, frequencies, and residual response for structure with five characteristic frequencies ($N_d = 4$, $n = 1$)

Design parameters		0.07132, 0.04079, 0.01906, 0.00779	
Residual mode	Natural freq, Hz	Final displacement	Final velocity
2	1.4303	-0.3×10^{-7}	-0.1×10^{-5}
3	3.4709	-0.9×10^{-9}	0.2×10^{-6}
4	7.4865	-0.3×10^{-9}	0.4×10^{-6}
5	17.4942	0.2×10^{-7}	-0.6×10^{-4}
6	24.3927	0.1×10^{-5}	-0.5×10^{-3}
7	42.0657	-0.2×10^{-5}	0.1×10^{-3}
8	72.1615	-0.8×10^{-6}	0.1×10^{-3}
9	136.5019	-0.4×10^{-7}	0.7×10^{-4}

Table 6 Parameters and lowest eight natural frequencies of four-boom structure ($n = 4$)

$N_d = 8$		$N_d = 16$	
Optimal parameters	Natural frequencies, Hz	Optimal parameters	Natural frequencies, Hz
0.07623	0.	0.08733	0.
0.04658	1.395	0.06572	1.410
0.03657	2.444	0.05220	2.453
0.02744	4.504	0.04260	4.453
0.01831	8.887	0.03500	7.010
0.01070	15.177	0.02879	10.931
0.00638	22.871	0.02371	16.148
0.00200	28.220	0.01952	23.118
		0.01620	
		0.01357	
		0.01118	
		0.00877	
		0.00616	
		0.00364	
		0.00200	
		0.00085	

is used as an initial guess for the iterative solution of the desired problem with N_d parameters.

V. Conclusions

The concept of a characteristic elastic system for a time-limited optimal maneuver has been discussed. The concept may or may not be useful in practice. Nevertheless, it is fundamentally important because it allows a physical understanding of the interaction between a flexible structure and its optimal time-limited control. The concept is also particularly satisfying because of its simplicity. In particular, an infinite number of characteristic elastic systems exist and one need not choose the one with the lowest mass, even though it is probably the one of most practical interest. Moreover, it has been graphically illustrated that the ratio of actual optimal control cost to optimal rigid-body control cost, while unity at specific points for a flexible structure, approaches unity everywhere as the structure is stiffened. Thus, the concept of a characteristic elastic system is consistent with the fact that flexibility effects are relatively unimportant when maneuvering stiff structures.

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